Part I: Intro

Today may be a bit disjointed.
We're trying to squeeze a lot of formalism about categories into one lecture so that we can do more concrete stuff toward the main theorem.

The current situation:

$$
\begin{tikzcd}
\mathrm{Lie} \arrow[rr, bend left, "\Psi"] \arrow[dr, "C^*_{\mathrm{Lie}}"] \arrow[ddr, bend right, "\Theta = \text{forget}"'] & & \mathrm{FMP} \arrow[ddl, bend left, "\mathbb{T}[-1] = -\circ \mathrm{Dual}"] \\
& (\mathrm{CAlg}^{\mathrm{aug}})^{op} \arrow[ur, "\mathrm{Spf}"] \arrow[d, "?"] & \\
& \mathrm{Mod}_k &
\end{tikzcd}
$$

Want: $\mathfrak{D} : (\mathrm{CAlg}^{\mathrm{aug}})^{op} \longrightarrow \mathrm{Lie}$

$\Psi^{-1} : \mathrm{FMP} \longrightarrow \mathrm{Lie}$

Q: How to show Ψ is an equivalence?

Idea: Find an "inverse"!

Standard strategy

Given $F: \mathcal{C} \to \mathcal{D}$, then

① find an adjoint $G: \mathcal{D} \to \mathcal{C}$

Recall one useful definition

Def A functor $F: \mathcal{C} \to \mathcal{D}$ is left adjoint to a functor $G: \mathcal{D} \to \mathcal{C}$ if there exists a natural transformation

$$\eta : id_{\mathcal{C}} \Rightarrow G \circ F$$

["unit of adjunction"]

such that for any $c \in \mathcal{C}$, $d \in \mathcal{D}$, and $f: c \to G(d)$ in $\mathcal{C}$, there exists a unique $g: F(c) \to d$ in $\mathcal{D}$ such that

$$f = G(g) \circ \eta(c)$$

$$\begin{tikzcd} c \arrow[r, "\eta(c)"] \arrow[rr, bend right, "f"'] & G(F(c)) \arrow[r, "G(g)"] & G(d) \end{tikzcd}$$

also exists a counit $\varepsilon: F \circ G \Rightarrow id_{\mathcal{D}}$

②

② show that

$$\eta : id_{\mathcal{C}} \Longrightarrow G \circ F \quad \& \quad \varepsilon : F \circ G \Longrightarrow id_{\mathcal{D}}$$

are isomorphisms

Looking for an adjoint feels easier & sometimes is obvious, even if the fact that it's an equivalence is not.

It also gives you a useful test criterion to see if your idea is plausible.

Prop A left adjoint preserves all colimits.
A right adjoint preserves all limits.

The proof is fun. If you don't know it, look it up!

There's a powerful partial converse.

Adjoint functor theorem(s)

if a functor $G : \mathcal{D} \to \mathcal{C}$ preserves limits
and satisfies [SMALLNESS CONDITION], several variants
then it has a left adjoint.

So the rest of the talk will be devoted to:

1. Learning more about colimits so we can prove (later!)

   Prop C^*_{Lie} preserves colimits.

   You analyze a small list of kinds of colimits that generate them all

2. Learning what an adjunction of ∞-categories is & what "smallness condition" there provides an ∞-categorical adjoint functor theorem